The joint measurement problem*

Jos Uffink
Foundations of Science
Universiteit Utrecht, P.O. Box 80.000
3508 TA Utrecht, the Netherlands

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Abstract

According to orthodox quantum theory, the joint measurement of non-commuting observables is impossible. It has been claimed recently that such joint measurements are admitted in a generalized formalism for quantum theory developed by Ludwig and Davies, by means of so-called 'unsharp observables'. It is argued in this paper that this claim has not been substantiated.

1 Introduction

Is it possible to measure both the position and the momentum of a particle? And if not, is it perhaps still possible to measure such quantities inaccurately? Questions like these form instances of what I shall call the 'joint measurement problem'. The received view on this problem is that quantum theory excludes the joint measurement of position and momentum. In fact, it is also denied that it is possible to attribute exact values of position and momentum to a particle as coexisting properties. However, the debate on this issue is still far from being settled.

One of the main reasons why this conclusion from quantum theory seems puzzling is that on a macroscopic scale one can obtain knowledge about the position and momentum of a particle simultaneously. It is often said that when the momentum of a particle is measured, we know at least that the particle is in the laboratory. Obviously, some argument is needed to reconcile this conflict between what is theoretically impossible and what yet appears to be done in the laboratory. The usual argument runs as follows. Real experiments are always inaccurate in some sense. Thus they should be described as measurements of a quantity involving a finite inaccuracy. Now, although simultaneous *exact* measurements of position and momentum are impossible, an inaccurate measurement of both quantities may still be possible. A limit on the inaccuracies of such a measurement is given by the well-known uncertainty relation $\Delta p \Delta q \geq \frac{1}{2}\hbar$. The conflict with actual experiment is then overcome by pointing out that $\hbar$ is so small that the restrictions quantum theory places on such simultaneous measurements can be neglected on the macroscopic scale.

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A weakness of this argument is that the uncertainty relation, in its usual formulation, refers to the expected spread (root-mean-square fluctuations) in separate (exact) measurements of position and momentum respectively. These do not represent the *inaccuracy* of the measurements, nor do they refer to joint measurements. In fact the formalism of quantum theory, as it is presented by Von Neumann, simply has no room for a description of a joint measurement of position and momentum at all.

In the last two decades a new approach to this problem has been proposed by a great number of authors, in particular by Prugovecki, Holevo, Busch, Lahti and Schroek and Martens and De Muynck. The starting point of this approach is a generalized formalism of quantum theory, developed by Ludwig (1976) and Davies (1976). It is claimed that within this generalized formalism non-commuting quantities like position and momentum can be jointly measured if a certain provision is made with respect to the sharpness or accuracy of these measurements. Thus one speaks of 'unsharp' or 'inaccurate' measurements, or measurements of 'unsharp' (or 'fuzzy','stochastic', 'approximate', etc.) quantities. It is the purpose of the present article to review this approach and examine its relevance to the joint measurement problem.

We shall find that the formalism of Ludwig and Davies does not yield new conclusions for this problem. In fact we shall conclude that the claim that within this formalism a joint unsharp measurement of position and momentum or of a pair of spin components is possible is false. We shall argue that this claim rests on the adoption of inappropriate definitions, i.e. definitions that trivialize the problem.

2 Joint measurements in the orthodox formalism

We review the basic aspects of the orthodox formalism of Von Neumann as far as relevant for joint measurements. It is postulated in this formalism that every observable quantity is represented by a self-adjoint operator on a Hilbert space $\mathcal{H}$. By virtue of the spectral theorem, every self-adjoint operator A has a unique spectral decomposition

$$A = \sum_i a_i A_i \tag{1}$$

where a_i denote its distinct eigenvalues and A_i its eigenprojections. (For simplicity, it is assumed that A has a finite spectrum. The projections A_i may be multidimensional.) The eigenprojections satisfy

$$\sum_i A_i = I\!I \quad , \quad A_i A_j = \delta_{ij} A_i \tag{2}$$

where $I\!I$ is the unit operator on $\mathcal{H}$. We call a collection $\{A_1, \ldots, A_n\}$ of projections obeying (2) a *spectral resolution.* Further, it is postulated that the probability of finding the value a_i when A is measured on a system in state ρ is $\mathrm{Prob}_\rho(a_i) = \mathrm{Tr}\rho A_i$.

An alternative approach to the orthodox formalism would be, in Mackey's words, to 'turn the spectral theorem on its head'. That is, one may also start from the postulate that every observable is represented by a mapping

$$\mathcal{A} : a_i \longrightarrow A_i$$

from a set $X^A = \{a_1, \dots, a_n\}$ of real values to a spectral resolution and invoke the spectral theorem to show that these mappings are in one-to-one correspondence with self-adjoint operators. These two approaches are equivalent in the present case. But, as we shall see, they are not equally easy to generalize.

The notion of joint measurements can be introduced formally as follows

Definition 1 *Two observables A, B are jointly measurable if there is a third observable C of which they are both functions, i.e. if $A = f(C)$ and $B = g(C)$.*

We also say that a measurement of any C with the above properties *is* in fact a joint measurement of A and B. The motivation for this definition is that if we measure the observable C, we can assign values to A and B simply by applying the functions f and g to the outcome c. Note that the above definition of joint measurement is not peculiar to quantum theory. It is a useful criterion also in a general class of physical theories including classical physics (Varadarajan, 1962).

If we think of observables as spectral mappings we can reformulate the definition as the following criterion: two observables $\mathcal{A}, \mathcal{B}$ are jointly measurable if there is a third observable $\mathcal{C}$, with spectral resolution $\{C_1, \dots, C_m\}$, such that

$$A_i = \sum_{k \in K_i} C_k \quad , \quad B_j = \sum_{k \in K'_j} C_k \tag{3}$$

Where $\{K_i\}$ and $\{K'_j\}$ denote two partitions of the index set $\{1, \dots, m\}$. That the definition entails the criterion can be seen by writing

$$K_i = \{k \mid a_i = f(c_k)\} \quad , \quad K'_j = \{k \mid b_j = g(c_k)\} \tag{4}$$

Conversely, for any two given partitions of $\{1, \dots, m\}$ one can always find functions f and g such that (4) holds. The general definition of commutativity is:

Definition 2 *Two observables $\mathcal{A}, \mathcal{B}$ commute iff $\forall i, j : \ A_i B_j = B_j A_i$*

With these definitions one obtains the well-known theorem:

Theorem 1 *Two observables are jointly measurable iff they commute.*

Joint measurements as defined here are often called *simultaneous* measurements. But there is no reason to assume that a joint measurement consists of two measurements performed simultaneously, i.e. at the same instant of time. In order to avoid connotations with simultaneity, Ludwig proposed the term ‘coexistent’ for jointly measurable observables. However, this term has other unwanted connotations. In ordinary language the statement “A and B are coexistent” is equivalent to: “both A and B are existent”. So, if A and B are coexistent and B and C are also coexistent, it is hard to refrain from the belief that A and C must also be coexistent. However, in quantum theory, commeasurability is not a transitive relation and the conclusion would be false in general. We therefore prefer to use the term ‘jointly measurable’.

3 Joint measurements in the generalized formalism

The most important distinction between the formalism of Ludwig and Davies and the orthodox formalism is that the notion of a spectral resolution is replaced by the notion of a so-called *semi-spectral resolution.* Restricting ourselves again to the finite case, we define a semi-spectral resolution as a collection of positive operators (also called *effects*) $\{M_1, \ldots, M_n\}$ on $\mathcal{H}$ such that

$$M_i \geq 0, \quad \sum_i M_i = I\!I \tag{5}$$

It is now postulated that an observable is represented by a mapping $\mathcal{M} : m_i \longrightarrow M_i$ from a spectrum X^M, which represents the set of possible values, to a semi-spectral resolution. Also, it is postulated that the probability of obtaining the value m_i in a measurement of the observable $\mathcal{M}$ on a system prepared in the state ρ is given by

$$\text{Prob}(m_i) = \text{Tr}\rho M_i$$

Obviously, one has $\text{Tr}\rho M_i \geq 0$ and $\sum_i \text{Tr}\rho M_i = 1$. Thus, the formalism provides a consistent generalization of the orthodox formalism. The latter is recovered as the special case in which the operators M_i are orthogonal projection operators. An important difference with the orthodox formalism is, however, that there is no analogous 'semi-spectral theorem' that would enable us to characterize an observable uniquely by a self-adoint operator (Grabowski 1989). Thus the two ways of introducing the notion of an observable of the previous section are no longer equivalent here.

Before considering this formalism in more detail, it is worthwhile giving some examples of semi-spectral resolutions to see what kind of new observables this formalism allows.

Examples

1. On any Hilbert space, the set $\{\frac{1}{2}I\!I, \frac{1}{2}I\!I\}$ constitutes a semi-spectral resolution. A measurement of this trivial observable is realized by ignoring the system altogether and tossing a fair coin.

2. Let $\mathcal{H} = \mathcal{C}^2$ be the Hilbert space of a spin-$\frac{1}{2}$ particle. Let $\vec{n}_1, \ldots \vec{n}_m$ be unit vectors in $I\!R^3$ such that $\sum_i \vec{n}_i = 0$, and $P_i = \frac{1}{2}(I\!I + \vec{n}_i \cdot \vec{\sigma})$ be the projectors for spin up in direction $\vec{n}_i$. Then
$$\left\{ \frac{2}{m} P_1, \ldots, \frac{2}{m} P_m \right\}$$
forms a semi-spectral resolution (cf. Holevo 1982).

3. On $\mathcal{C}^2$, let P_+, P_- be the eigenprojections of spin in the z-direction. Then
$$\{(1 - \epsilon)P_+ + \delta P_- \ , \ \epsilon P_+ + (1 - \delta)P_-\}$$
forms a semi-spectral resolution. One can think of this as the representation of a measurement of spin in the z-direction where sometimes (with probability ϵ) a 'spin-up' result is registered as 'down', and particles with spin-down are sometimes (probability δ) registrated as 'up'. (cf. Busch and Schroeck 1989)

From these examples we see that the operators M_i which constitute a semi-spectral resolution need not be projectors, nor need they commute with each other (example 2). Also, the number of such operators need not be bounded by the dimension of $\mathcal{H}$.

We define the notions of joint measurements and commutativity in analogy with the orthodox formalism:

Definition 3 *Two observables $\mathcal{M}, \mathcal{N}$ are jointly measurable iff there is a third observable $\mathcal{O}$ such that*

$$M_i = \sum_{k \in K_i} O_k \quad N_j = \sum_{k \in K'_j} O_k$$

Definition 4 *Two observables $\mathcal{M}, \mathcal{N}$ commute iff $\forall i, j : \ M_i N_j = N_j M_i$.*

Let us now reconsider the joint measurement problem. We first discuss the observables of the orthodox formalism, which, as we have seen, are imbedded in the new formalism. We call these the 'orthodox observables'. Let $\mathcal{A}, \mathcal{B}$ be orthodox observables and suppose that they were not jointly measurable in the orthodox formalism. Is this true also in the new formalism? The answer is not *a priori* obvious. Whether observables are jointly measurable depends on how many observable quantities are admitted by the theory. And since the new formalism is more liberal than the orthodox formalism, it is possible in principle that $\mathcal{A}$ and $\mathcal{B}$ do become jointly measurable in the new theory. However, it can be shown:

Theorem 2 *A pair of orthodox observables is jointly measurable iff they commute.*

(Cf. Ludwig 1976, Davies 1976) Thus we recover the same conclusion as in Von Neumann's formalism. Our only possibility for new results therefore lies in a consideration of unorthodox observables, i.e. observables associated with semi-spectral resolutions. Then one can show

Theorem 3 *Commutativity of observables is a sufficient but not necessary condition for joint measurability.*

The proof of this theorem is so simple that we reproduce it.

> Proof. If $\mathcal{M}$ commutes with $\mathcal{N}$ the mapping $\mathcal{O} : (i, j) \longrightarrow O_{ij} = M_i N_j$ defines an observable whose measurement provides a joint measurement of $\mathcal{M}$ and $\mathcal{N}$. That the condition of commutatitvity is not necessary follows from the fact that any $\mathcal{M}$ is jointly measurable with itself, but need not commute with itself.

Thus, unorthodox observables can be jointly measurable, even if they don't commute. At first sight, this looks like an exciting new result. But since the proof is so simple, it is not clear whether something significant is at stake. In any case, in order to judge the relevance of the fact that non-commuting unorthodox observables can be jointly measurable we should consider in more detail what is the meaning of these unorthodox observables.

4 Unsharp observables

We have seen that the Ludwig-Davies formalism is more liberal than the orthodox formalism of Von Neumann. It allows the measurement of what we have called 'unorthodox observables'. But we have not yet discussed the interpretation of the unorthodox observables. One would naturally like to know *what* is being measured in a measurement of an unorthodox observable. Here we discuss some possible answers to this question.

The first option is to think of unorthodox observables as corresponding to new physically meaningful quantities not recognized by von Neumann's formalism. In some cases specific interpretations for interesting new quantities have been proposed. Examples are Lévy-Leblond's proposal (1976) for an observable representing the phase of an harmonic oscillator , or Holevo's proposal (1982) for a time observable. Both instances are notoriously problematic from an orthodox viewpoint. However, it is not clear how one should proceed in general.

Of course one can argue that this question is to be decided in the context of concrete applications, or by extra-theoretical considerations, just as the analogous question in the orthodox formalism (which physical quantity is represented by which self-adjoint operator?) is likewise not decided by the formalism itself. This is a legitimate argument, but not very helpful in our case. As we have seen in the previous section, the new formalism gives the 'old' answer to the 'old' joint measurement problem. It only gives a new answer to a new problem, viz. the joint measurement of unorthodox observables. Unless we are able to say more about the meaning of these new observables, or about how they are related to the orthodox observables, we have gained no new insight in the problem with which we started.

A next proposal consists of introducing a pre-ordering relation between observables having the purported meaning that one observable is an 'unsharp' 'fuzzy' 'stochastic' or 'inaccurate' version of another. This relation was introduced and studied by She and Heffner (1966) Ali and Emsch (1974), Prugovecki (1976), Busch (1985) and others. The following definition is essentially due to Martens and De Muynck (1990a):

Definition 5 *The observable $\mathcal{M}$ is called an unsharp version of $\mathcal{N}$ iff $N \succ M$, where $\succ$ is defined as follows:*

(a) $\mathcal{N} \succeq \mathcal{M}$ iff there is a stochastic matrix (λ_{ki}) (i.e. a matrix with $\lambda_{ki} \geq 0$ and $\sum_k \lambda_{ki} = 1$) such that

$$M_k = \sum_i \lambda_{ki} N_i \tag{6}$$

(b) $\mathcal{N} \succ \mathcal{M}$ iff $\mathcal{N} \succeq \mathcal{M}$ and $\mathcal{M} \not\succeq \mathcal{N}$.

We also say in this case that a measurement of $\mathcal{M}$ is an unsharp measurement of $\mathcal{N}$. The motivation for this definition can easily be explained. Suppose a measurement procedure is conducted as follows. Actually, a measurement of $\mathcal{N}$ is performed and a result n_i is obtained, but due to internal noise in the detector or some other random process, this result is not properly registrated as the outcome of the experiment. Instead, whenever the result n_i has been obtained, the outcome m_k is recorded with conditional probability $p(m_k|n_i) = \lambda_{ki}$. In this situation, for any state ρ, the probability of recording the value m_k is given by

$$\mathrm{Prob}_\rho(m_k) = \sum_i \lambda_{ki} \mathrm{Prob}_\rho(n_i) = \sum_i \lambda_{ki} \mathrm{Tr} \rho N_i = \mathrm{Tr} \rho M_k$$

Hence, this procedure is (equivalent to) a measurement of the observable $\mathcal{M}$.

Now, of course, the suggested interpretation is by no means unique, and indeed, the multitude of names by which the relation of definition 5 is known reflects the divergent interpretational tastes of the authors who have contributed to this subject. It is not necessary, however, to discuss this issue in the context of this paper, because the mathematical formulation of definition 5 is indifferent to this aspect of the problem of interpretation. (Although some would perhaps allow λ_{ki} to depend on the state of the system or other circumstances.)

Definition 5 gives an articulation of the idea of a noisy or inaccurate measurement. The important point here is that in spite of such noise, a measurement of $\mathcal{M}$ generally yields useful information about $\mathcal{N}$. When a value m_i is obtained, one can make a statistical inference about the values of $\mathcal{N}$, in the manner of estimation theory, confidence intervals or some other statistical technique (cf. Busch and Schroeck 1989). Thus, measuring $\mathcal{M}$ gives 'partial information' about $\mathcal{N}$.

For orthodox observables one has $\mathcal{A} \succ \mathcal{B}$ iff $B = f(A)$ when f is a non-bijective function. Thus the relation "$\succ$" generalizes the notion of one observable being a function of another. It reduces to the latter notion when the stochastic matrix λ contains only zeroes and ones. Also note that one observable $\mathcal{M}$ may be an unsharp version of several observables $\mathcal{N}, \mathcal{L}, \ldots$ which need not be jointly measurable.

Of particular interest are those observables which are not unsharp versions of other observables. These are the analogues of the maximal observables in Von Neumann's formalism, and it seems appropriate to call them the 'sharp' observables. It is found (Martens and De Muynck 1990) that an observable is sharp iff its semi-spectral resolution is of the form $\{\alpha_1 P_1, \ldots, \alpha_n P_n\}$ where P_i denote one-dimensional projectors (not necessarily commuting!) and $0 < \alpha_i \leq 1$. All orthodox maximal observables are sharp in this sense. Example 2 above shows that there are also sharp unorthodox observables.

It should be mentioned that there also are two other definitions of 'unsharpness' current in the literature which are quite different from definition 5. One proposal is to go back to the orthodox idea that the physically meaningful quantities are represented by self-adjoint operators and use a semi-spectral decomposition

$$A = \sum_i m_i M_i \tag{7}$$

to link them with unorthodox observables. In this option one regards an observable $\mathcal{M}$ as an unsharp version of an orthodox observable $\mathcal{A}$ in case (7) holds. (Cf. Ali and Doebner 1976, Holevo 1982, Schroeck 1985 and Schroeck 1989). Grabowski (1989) has conjectured that the two definitions are equivalent (in the case when they are both applicable, i.e. when $\mathcal{N}$ in definition 5 is orthodox). This conjecture, however, is false.[1]

The present proposal faces some problems. There are semi-spectral resolutions that decompose *any* self-adjoint operator on a given Hilbert space (by choosing the values m_i appropriately). In the present option this would mean that one can measure *all* observables of a system in a single experiment, merely by relabeling the outcomes. This would, indeed, offer a radical new solution to the joint measurement problem. But is it acceptable?

As an example, consider a spin-$\frac{1}{2}$ particle and a random device with three equally likely possible outcomes (say a common die with outcomes {1,2}, {3,4}, {5,6}). Throw the die and decide, depending on the result of this throw, to measure either σ_x, σ_y or σ_z. The semi-spectral decomposition for this procedure is $\{\frac{1}{3}P_{x+}, \frac{1}{3}P_{x-}, \frac{1}{3}P_{y+}, \ldots, \frac{1}{3}P_{z-}\}$, where $\{P_{i+}, P_{i-}\}$ are the spectral resolutions of σ_i. Suppose this procedure is performed and we obtain the outcome $k = 1$. That is, the die gave the result 1 or 2, we measured σ_x and

[1] Every observable $\mathcal{M}$ with $\mathcal{A} \succ \mathcal{M}$ ($\mathcal{A}$ orthodox) commutes with itself. This is not the case for every semi-spectral resolution of A.

obtained spin up. Is this justification enough to say that we have measured all observables of the system? That would seem preposterous.

Another motivation for regarding the above procedure as a measurement of all observables is, perhaps, that it is 'informationally complete'. This means that from the measurement statistics one can estimate the precise form of ρ. In our example this arises because, when the procedure is repeated, one will eventually also measure σ_y and σ_z. Thus the 'complete information' is obtained only in a *sequence* of measurements on the same state ρ. But that is not really a joint measurement. The same information is obtained (and actually more efficiently) if the die throw is deleted from the procedure and one decides rightaway to perform a sequence of orthodox σ_x, σ_y and σ_z measurements.

Yet another definition of unsharpness is proposed by Busch (1985, 1986a), Busch and Schroeck (1989). Here, an observable is said to be unsharp if its range contains an operator M_k such that for some state ρ one has $\mathrm{Tr}\rho M_k > \frac{1}{2}$ and for some other state ρ' one has $\mathrm{Tr}\rho' M_k < \frac{1}{2}$. This definition differs from both of the previous proposals. For example, in contrast to those proposals, one should now say that all orthodox observables are unsharp, while $\{\frac{1}{2}I\!I, \frac{1}{2}I\!I\}$ is not. But the foremost difference seems to be that here the term 'unsharp' is treated as a quality of observables, ("observable $\mathcal{M}$ is unsharp") and not as a relationship between them ("observable $\mathcal{M}$ is an unsharp version of $\mathcal{A}$"). In view of this, the present proposal does not seem helpful for the interpretation of unorthodox observables. Our problem is not which observables are to be called unsharp, but rather what they stand for, i.e. *what* is being unsharp.

Studying this last question, one finds that certain unorthodox observables are identified with 'unsharp momentum', 'unsharp spin', etc. (Busch 1985, 1985a, 1987, Busch and Schroeck 1989.) But this makes sense only if one assumes some particular relationship between these observables and the orthodox momentum, spin etc. observables. It seems that in all cases this relationship is assumed to be of the form (6). This, in effect, leads us back to definition 5. To summarize, there are three different ways in which a notion of unsharpness is introduced in the literature, only one of which (definition 5) seems useful for our purpose.

Let us now return to the joint measurement problem. We have seen in the previous section that the generalized formalism leads to new conclusions with respect to the joint measurement problem for unorthodox observables only. We have now seen (by definition 5) how certain unorthodox observables can be interpreted as unsharp versions of orthodox observables. So the question arises whether non-commuting orthodox observables become jointly measurable if we replace them by unsharp versions. Consider the following definition: (Martens 1991)

Definition 6 *An observable $\mathcal{O}$ is said to be a joint unsharp version of $\mathcal{A}$ and $\mathcal{B}$ if there exist observables M and N such that*

$$M_i = \sum_{k \in K_i} O_k \quad , N_j = \sum_{k \in K'_j} O_k \tag{8}$$

and

$$A \succ M \quad B \succ N \tag{9}$$

In this case we also say that a measurement of $\mathcal{O}$ is a joint unsharp measurement of $\mathcal{A}$ and $\mathcal{B}$.

Before we discuss this definition further, we give an example: joint unsharp position and momentum (skipping the mathematical details concerning the continous spectrum of position and momentum).

Example 4 Let $\mathcal{H} = \mathcal{L}^2(I\!R)$, $|\chi\rangle \in \mathcal{H}$ be any state vector and define $|\chi_{pq}\rangle = e^{i(qP-pQ)}|\chi\rangle$. where P and Q are the orthodox momentum and position operators, and p and q are real numbers. Then the operators

$$O(p,q) = (2\pi)^{-1}|\chi_{pq}\rangle\langle\chi_{pq}|$$

define a bivariate continuous analogue of a semi-spectral resolution. That is, we have

$$O(p,q) \geq 0 \quad , \quad \int\!\!\int O(p,q)\, dp\, dq = I\!I$$

The mapping $\mathcal{O} : (p,q) \longrightarrow O(p,q)$ is a joint unsharp version of position and momentum according to definition 6. Indeed the marginals of $O(p,q)$

$$M(p) = \int O(p,q)\, dq \quad , \quad N(q) = \int O(p,q) dp \tag{10}$$

are the continuous coarse-grained resolutions analogous to (8). At the same time

$$M(p) = \int |\langle\chi|p'-p\rangle|^2\, |p'\rangle\langle p'| dp' \quad , \quad N(q) = \int |\langle\chi|q'-q\rangle|^2\, |q'\rangle\langle q'| dq' \tag{11}$$

are smoothened versions of the ordinary momentum and position resolutions, analogous to (9). Here $|\langle\chi|p'-p\rangle|^2 = \lambda(p,p')$ and $|\langle\chi|q'-q\rangle|^2 = \mu(q,q')$ replace the stochastic matrices in (6). The joint probability density $\mathrm{Tr}\rho O(p,q)$ is called the 'smoothened' Wigner distribution, the 'stochastic phase space' representation or Husimi representation of ρ. (She and Heffner 1966, Prugovecki 1984, Busch 1985a, 1987b, Ali 1985, Braunstein, Caves and Milburn 1991)

In general, the motivation behind definition 6 may be put as follows. If there are observables $\mathcal{M}, \mathcal{N}$ obeying (9), these are unsharp versions of $\mathcal{A}$ and $\mathcal{B}$, according to definition 5. Thus, a measurement of $\mathcal{M}$ gives some 'unsharp information' about $\mathcal{A}$, and likewise for $\mathcal{N}$ and $\mathcal{B}$. Moreover, (8) states that $\mathcal{M}$ and $\mathcal{N}$ are jointly measurable through $\mathcal{O}$. Hence, if we measure $\mathcal{O}$, we realize an unsharp measurement of $\mathcal{A}$ as well as of $\mathcal{B}$.

This argument may appear tempting, but it is incorrect. The conclusion that a measurement of $\mathcal{O}$ is an unsharp measurement of both $\mathcal{A}$ and $\mathcal{B}$ would mean, according to our previous definitions, that $\mathcal{A} \succ \mathcal{O}$ and $\mathcal{B} \succ \mathcal{O}$. But clearly, neither of these relations is implied by the above definition.

The pitfall in the above argument is that joint measurability is not a transitive relationship. This becomes perhaps more transparent if we first note that definition 6 can also be implemented in the orthodox formalism. That is, one can take all the observables mentioned in this definition to be orthodox. For example, consider a particle with spin and one spatial degree of freedom and let

$$\begin{array}{ccccc} \mathcal{A} = (\sigma_x, Q) & & \mathcal{O} = (\sigma_x, P) & & \mathcal{B} = (\sigma_y, P) \\ & \searrow\swarrow & & \searrow\swarrow & \\ & \mathcal{M} = \sigma_x & & \mathcal{N} = P & \end{array}$$

where arrows represent the preordering $\succeq$.

Now, an experimenter measuring $\mathcal{M}$ (spin in the x-direction) might say to himself "If only I had used a more accurate instrument, I could have measured $\mathcal{A}$ (spin and position)." This counterfactual belief will not lead to difficulties, just because $\mathcal{M}$ and $\mathcal{A}$ are jointly measurable (or 'coexistent'). In a similar way the experimenter is entitled to say: "If I had used another more accurate instrument, measuring $\mathcal{O}$, I could have obtained the values of spin σ_x as well as momentum P." But $\mathcal{A}$ and $\mathcal{O}$ are not jointly measurable. Hence if the experimenter actually sets up an instrument to measure $\mathcal{O}$, he is no longer justified in believing that he could still also have measured $\mathcal{A}$ without sacrifice of his actual results, let alone in saying that he has in fact measured $\mathcal{A}$ inaccurately or unsharply. A similar argument holds for $\mathcal{B}$.

Thus, the above definition does not guarantee that a measurement of $\mathcal{O}$ is an unsharp measurement of either $\mathcal{A}$ or $\mathcal{B}$. In fact $\mathcal{O}$ need not even be jointly measurable with $\mathcal{A}$ or $\mathcal{B}$. To call such observables a joint unsharp version of $\mathcal{A}$ and $\mathcal{B}$ is to use the term 'joint' in a way that can no longer be equated with 'jointly' or 'both'. The danger of this pitfall is illustrated by the fact that some authors (Busch 1987a , Martens 1991) apply the above definition to discuss the question whether in an interference experiment one can observe *both* the path of a particle and the interference phenomenon.

Furthermore, note that by definition 6 the measurement of any observable whatsoever is a joint unsharp measurement of an arbitrary pair of observables. (E.g. take the range of $\mathcal{M}$,$\mathcal{N}$ equal to $\{\mathbb{I}\}$. Thus definition 6 has no counterexamples. This conclusion can of course be blocked by adding further restrictions to the definitions 5 and 6. For example, some authors demand that the stochastic matrix in definition 5 is symmetrical. Under such constraints the notion of joint unsharp measurements becomes non-trivial, but the constraints themselves are somewhat ad hoc. However this may be, it is the first point raised here which to me seems to be the most serious objection to definition 6. It seems inappropriate to define joint unsharp measurements in such a way that a 'joint unsharp measurement of $\mathcal{A}$ and $\mathcal{B}$' is different from a procedure in which both $\mathcal{A}$ and $\mathcal{B}$ are measured unsharply. Let me end with a remark. The above criticism of definition 5 should not be construed as meaning that the 'stochastic phase-spase' or 'coherent-state' observable of example 4 is trivial or useless. Rather, it means that its interest lies elsewhere. Constructions like example 4 are today often used in quantum optics (Klauder and Skagerstam 1985). In these applications the 'position' and 'momentum' operators are actually the real and imaginary part of the quantized field amplitude (of a single mode). This suggests that it may be more fruitful to interpret this observable not as a 'joint unsharp position and momentum', but as a (sharp!) observable representing the complex field amplitude.

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